Linear Algebra II

06/05/2014, Monday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 (15 pts) **Gram-Schmidt process**

Consider the vector space $\mathbb{R}^3$ with the inner product

$$\langle x, y\rangle = x^T y.$$

Apply the Gram-Schmidt process to transform the basis

$$\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \}$$

into an orthonormal basis.

REQUIRED KNOWLEDGE: **inner product, Gram-Schmidt process**

SOLUTION:

To apply the Gram-Schmidt process, let

$$x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad x_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} \quad x_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

and note that

$$\begin{array}{ccc} \langle x_1, x_1\rangle = 3 & \langle x_1, x_2\rangle = 2 & \langle x_1, x_3\rangle = 1 \\ & \langle x_2, x_2\rangle = 2 & \langle x_2, x_3\rangle = 1 \\ & & \langle x_3, x_3\rangle = 1. \end{array}$$

By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|} \qquad p_1 = \langle x_2, u_1 \rangle u_1 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_2 - p_1 = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\|x_2 - p_1\|^2 = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{2}{3}$$

$$\|x_2 - p_1\| = \frac{\sqrt{2}}{\sqrt{3}}$$

$$u_2 = \frac{\sqrt{3}}{\sqrt{2}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|} \qquad p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x_3 - p_2 = \frac{1}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\|x_3 - p_2\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\|x_3 - p_2\| = \frac{1}{\sqrt{2}}$$

$$u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

By using Cayley-Hamilton theorem, show that $M^k = 2^{k-2}M^2$ for any $k \geqslant 2$.

REQUIRED KNOWLEDGE: **Cayley-Hamilton theorem**

SOLUTION:

Cayley-Hamilton theorem states that $p_M(M) = 0$ where p_M denotes the characteristic polynomial of M, that is $p_M(\lambda) = \det(M - \lambda I)$. Note that

$$\begin{aligned} p_M(\lambda) = \det(M - \lambda I) &= \det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ -1 & 1-\lambda & 1 \\ 0 & 2 & -\lambda \end{bmatrix}\right) \\ &= -2\det\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1 \end{bmatrix}\right) - \lambda\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) \\ &= -2(1-\lambda-1) - \lambda\left((1-\lambda)^2 + 1\right) = 2\lambda - \lambda(\lambda^2 - 2\lambda + 2) \\ &= \lambda(2 - \lambda^2 + 2\lambda - 2) = \lambda(-\lambda^2 + 2\lambda) = -\lambda^3 + 2\lambda^2. \end{aligned}$$

Therefore, Cayley-Hamilton theorem implies that

$$-M^3 + 2M^2 = 0 \quad\implies\quad M^3 = 2M^2.$$

Note that the claim holds for $k = 2$. Suppose that it holds for an integer $\ell \geqslant 2$, that is

$$M^\ell = 2^{\ell-2}M^2.$$

Multiplying by M yields

$$M^{\ell+1} = 2^{\ell-2}M^3 = 2^{\ell-2} \cdot 2 \cdot M^2 = 2^{\ell-1}M^2.$$

Then, it follows from induction that $M^k = 2^{k-2}M^2$ for all $k \geqslant 2$.

Consider the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix}.$$

(a) Find a singular value decomposition for M.

(b) Find the best rank 1 approximation of M.

REQUIRED KNOWLEDGE: **singular value decomposition, lower rank approximations.**

SOLUTION:

(3a): Note that

$$M^T M = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}.$$

Then, the characteristic polynomial of $M^T M$ can be found as

$$\begin{aligned} p_{M^T M}(\lambda) &= \det\left(\begin{bmatrix} 13-\lambda & 12 \\ 12 & 13-\lambda \end{bmatrix}\right) \\ &= (\lambda - 13)^2 - 12^2 = (\lambda - 25)(\lambda - 1). \end{aligned}$$

Then, the eigenvalues of $M^T M$ are given by

$$\lambda_1 = 25 \quad \text{and} \quad \lambda_2 = 1$$

and hence the singular values by

$$\sigma_1 = 5 \quad \text{and} \quad \sigma_2 = 1.$$

Next, we need to diagonalize $M^T M$ in order to obtain the orthogonal matrix V. To do so, we first compute eigenvectors of $M^T M$.

For the eigenvalue $\lambda_1 = 25$, we have

$$0 = \left(M^T M - 25I\right)x = \begin{bmatrix} -12 & 12 \\ 12 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This results in the following (normalized) eigenvector

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 1$, we have

$$0 = \left(M^T M - 9I\right)x = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This results, for instance, in the following eigenvector

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, we get

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Note that the rank of M is equal to the number of nonzero singular values. Thus, $r = \operatorname{rank}(M) = 2$. By using the formula

$$u_i = \frac{1}{\sigma_i} M v_i$$

for $i = 1, 2$, we obtain

$$u_1 = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 3 \\ 3 \\ 4\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The last column vector of the matrix U can be found by looking at the null space of M^T:

$$\begin{bmatrix} 2 & 1 & 2\sqrt{2} \\ 1 & 2 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0.$$

This yields (the normalized) solution

$$u_3 = \frac{1}{5\sqrt{2}} \begin{bmatrix} -2\sqrt{2} \\ -2\sqrt{2} \\ 3 \end{bmatrix}.$$

Finally, the SVD can be given by:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{2}{5} \\ \frac{3}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{2}{5} \\ \frac{4}{5} & 0 & \frac{3}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(3b): The best rank 1 approximation can be obtained as follows:

$$\tilde{M} = \begin{bmatrix} \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{2}{5} \\ \frac{3}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{2}{5} \\ \frac{4}{5} & 0 & \frac{3}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \\ 2\sqrt{2} & 2\sqrt{2} \end{bmatrix}.$$

Let $J \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix, that is $J + J^T = 0$.

(a) Show that the real part of any eigenvalue of J must be zero.

(b) Show that J is singular if n is an odd number.

(c) Show that J is unitarily diagonalizable.

(d) Show that if λ is an eigenvalue of J then $\sqrt{-\lambda^2}$ is a singular value of J.

(e) Show that if σ is a singular value of J then $i\sigma$ is an eigenvalue of J.

REQUIRED KNOWLEDGE: **eigenvalues and singular values.**

SOLUTION:

(4a): Let (λ, x) be an eigenpair of J, that is

$$Jx = \lambda x.$$

Note that

$$\lambda x^H x = x^H Jx = -x^H J^T x = -(x^H Jx)^H.$$

It follows from $x^H Jx = -(x^H Jx)^H$ that the real part of $x^H Jx$ is zero. Since

$$\lambda = \frac{x^H Jx}{x^H x}$$

and $x^H x$ is a positive number, the real part of λ must be zero.

(4b): On the one hand, we have

$$\det(J) = \det(J^T)$$

since determinant is invariant under transposition. On the other hand, we have

$$\det(J) = \det(-J^T) = (-1)^n \det(J^T)$$

since $J + J^T = 0$. In case n is odd, the last two equations hold if and only if $\det(J) = 0$.

(4c): A matrix M is unitarily diagonalizable if and only if $M^H M = MM^H$. Since $J^H = J^T = -J$, we see that $J^H J = JJ^H = -J^2$. As such, J is unitarily diagonalizable.

(4d): Let (λ, x) be an eigenpair of J, that is

$$Jx = \lambda x.$$

Then, we have

$$J^T Jx = -J^2 x = -\lambda Jx = -\lambda^2 x.$$

In other words, $-\lambda^2$ is an eigenvalue of $J^T J$. This means that $\sqrt{-\lambda^2}$ is a singular value of J.

(4e): If σ is a singular value of J, then σ^2 must be an eigenvalue of $J^T J = -J^2$. As observed in the previous subproblem, this is only possible if $\pm\sqrt{-\sigma^2}$ is an eigenvalue of J. Note that $\pm\sqrt{-\sigma^2} = \pm i\sigma$.

(a) Consider the function

$$f(x, y, z) = -\frac{1}{4}(\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{y^4}) + yz - x - 2y - 2z.$$

Show that $(1, 1, 1)$ is a stationary point and determine whether this stationary point is local minimum/maximum or saddle point.

(b) Let

$$M = \begin{bmatrix} a+1 & a & 0 \\ a & a+1 & a \\ 0 & a & a \end{bmatrix}$$

where a is a real number. Determine all values of a for which M is

(i) positive definite.

(ii) negative definite.

REQUIRED KNOWLEDGE: **stationary points, positive definiteness.**

SOLUTION:

(5a): In order to find the stationary points, we need the partial derivatives:

$$f_x = x^{-5} - 1 \quad f_y = y^{-5} + z - 2\text{and} \quad f_z = z^{-5} - 3x.$$

Then, $(\bar{x}, \bar{y})$ is a stationary point if and only if

$$3\bar{x}^2 - 3\bar{y} = 0$$
$$3\bar{y}^2 - 3\bar{x} = 0.$$

This leads to $\bar{x}^4 = \bar{x}$, or equivalently $\bar{x}(\bar{x}^3 - 1) = 0$. Hence, we have $\bar{x} = 0$ or $\bar{x} = 1$ since $\bar{x}^3 - 1 = (\bar{x} - 1)(\bar{x}^2 + \bar{x} + 1)$. Then, the stationary points are $(\bar{x}, \bar{y}) = (0, 0)$ or $(\bar{x}, \bar{y}) = (1, 1)$. To determine the character of these points, we need the second order partial derivatives:

$$f_{xx} = 6x, \quad f_{xy} = -3, \quad \text{and} \quad f_{yy} = 6y.$$

For the stationary point $(\bar{x}, \bar{y}) = (0, 0)$, we have

$$H_{(0,0)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}.$$

Note that the characteristic polynomial of the Hessian matrix $H_{(0,0)}$ is given by $\lambda^2 - 9$. Hence, it has one positive ($\lambda_1 = 3$) and one negative eigenvalue ($\lambda_2 = -3$). Therefore, $H_{(0,0)}$ is indefinite and the stationary point $(\bar{x}, \bar{y}) = (0, 0)$ is a saddle point.

For the stationary point $(\bar{x}, \bar{y}) = (1, 1)$, we have

$$H_{(1,1)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}_{(1,1)} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

Note that the characteristic polynomial of the Hessian matrix $H_{(1,1)}$ is given by $(\lambda - 6)^2 - 9$. Hence, it has two positive eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 3$. This means that $H_{(1,1)}$ is positive definite. Consequently, stationary point $(\bar{x}, \bar{y}) = (1, 1)$ corresponds to a local minimum.

(5b): A symmetric matrix is positive definite if and only if all its leading principal minors are positive. Note that the leading principal minors of M are given by:

$$\det(a), \quad \det\left(\begin{bmatrix} a & -a \\ -a & b \end{bmatrix}\right) = ab - a^2, \quad \text{and} \quad \det\left(\begin{bmatrix} a & -a & 0 \\ -a & b & a \\ 0 & a & a \end{bmatrix}\right) = a(ab - a^2) - a^3 = a^2 b - 2a^3.$$

Then, the matrix M is positive definite if and only if

$$a > 0, \quad ab - a^2 > 0, \quad \text{and} \quad a^2 b - 2a^3 > 0.$$

This is, however, equivalent to saying that

$$a > 0, \quad b - a > 0, \quad \text{and} \quad b - 2a > 0.$$

Note that the inequality in the middle is implied by the last one. Hence, we can conclude that the matrix M is positive definite if and only if

$$a > 0 \quad \text{and} \quad b > 2a.$$

Consider the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Find its eigenvalues.

(b) Is it diagonalizable? Why?

(c) Put it into the Jordan canonical form.

REQUIRED KNOWLEDGE: **eigenvalues/vectors, Jordan canonical form, diagonalization.**

SOLUTION:

(6a): Note that

$$\begin{aligned}
\det\left(\begin{bmatrix} -\lambda & 0 & -1 \\ -2 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{bmatrix}\right) &= -\lambda \det\left(\begin{bmatrix} 1-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix}\right) - \det\left(\begin{bmatrix} -2 & 1-\lambda \\ 1 & 0 \end{bmatrix}\right) \\
&= -\lambda(1-\lambda)(2-\lambda) + (1-\lambda) \\
&= (1-\lambda)(-2\lambda + \lambda^2 + 1) = (1-\lambda)^3.
\end{aligned}$$

Hence, we have $\lambda_{1,2,3} = 1$.

(6b): It is diagonalizable if and only if it has 3 linearly independent eigenvectors. To find the eigenvectors, we need to solve the equation:

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

This leads to $x_1 = x_3 = 0$. Thus, eigenvectors must be of the form

$$x = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This means that we can find only one linearly independent eigenvector. Therefore, the matrix is not diagonalizable.

(6c): Since there is only one linearly independent eigenvector, Jordan canonical form consists of one block. Note that

$$(A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A-I)^3 = 0.$$

Next, we check if

$$(A-I)^2 v = x$$

has a solution where x is an eigenvector. Note that

$$(A-I)^2 v = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 + v_3 \\ 0 \end{bmatrix}.$$

Hence, we have

$$(A - I)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This means that

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, (A - I) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, (A - I)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

would generate a cyclic subspace. Then, we have

$$\begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{J}.$$